

Length and multiplicity of the local cohomology with support in a hyperplane arrangement

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September 4, 2015

Abstract

Let R be the polynomial ring in n variables with coefficients in a field K of characteristic zero. Let D_n be the n -th Weyl algebra over K . Suppose that $f \in R$ defines a hyperplane arrangement in the affine space K^n . Then the length and the multiplicity of the first local cohomology group $H_{(f)}^1(R)$ as left D_n -module coincide and are explicitly expressed in terms of the Poincaré polynomial or the Möbius function of the arrangement.

1 Introduction

Let K be a field of characteristic zero and $R = K[x] = K[x_1, \dots, x_n]$ be the polynomial ring in n variables $x = (x_1, \dots, x_n)$. For a nonzero polynomial $f \in K[x]$, let us consider the first local cohomology group $H_{(f)}^1(R) = R[f^{-1}]/R$, where $R[f^{-1}] = R_f$ is the localization of R with respect to the multiplicative set $\{f^k \mid k \in \mathbb{N}\}$ with $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Let $D_n = R\langle \partial \rangle = R\langle \partial_1, \dots, \partial_n \rangle$ be the n -th Weyl algebra, i.e., the ring of differential operators with polynomial coefficients with respect to the variables x , where we denote $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i = \partial/\partial x_i$ being the derivation with respect to x_i . An arbitrary element P of D_n is written in a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha \quad \text{with} \quad a_\alpha(x) \in K[x],$$

where we denote $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

Then $H_{(f)}^1(R)$ has a natural structure of left D_n -module and is holonomic ([6]). We are mainly concerned with the length and the multiplicity of

$H_{(f)}^1(R)$ as a left D_n -module in case f defines a hyperplane arrangement \mathcal{A} in the affine space K^n ; i.e., f is a multiple of linear (i.e., first-degree) polynomials. In particular, we show that the length and the multiplicity both coincide with $\pi(\mathcal{A}, 1) - 1$, where $\pi(\mathcal{A}, t)$ is the Poincaré polynomial of the arrangement \mathcal{A} .

The length of $R[f^{-1}]$ as left D_n -module, which equals that of $H_{(f)}^1(R)$ plus one, with f defining a hyperplane arrangement was studied e.g., in [1], [8]. The characteristic cycle of the local cohomology with respect to an arrangement of linear subvarieties was studied in [2]. Although not explicitly stated, Corollary 1.3 of [2] should yield main results of this paper. We give a direct proof for hyperplane arrangements.

2 Length and multiplicity

First let us recall basic facts about the length and the multiplicity of a left D_n -module following J. Bernstein ([3]). Let M be a finitely generated left D_n -module. A composition series of M of length k is a sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

of left D_n -submodules such that M_i/M_{i-1} is a nonzero simple left D_n -module (i.e. having no proper left D_n -submodule other than 0) for $i = 1, \dots, k$. The length of M , which we denote by $\text{length } M$, is the least length of composition series (if any) of M . If there is no composition series, the length of M is defined to be infinite. The length is additive in the sense that if

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

is an exact sequence of left D_n -modules of finite length, then $\text{mult } M = \text{mult } N + \text{mult } L$ holds.

For each integer k , set

$$F_k(D_n) = \left\{ \sum_{|\alpha|+|\beta| \leq k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in K \right\}.$$

In particular, we have $F_k(D_n) = 0$ for $k < 0$ and $F_0(D_n) = K$. The filtration $\{F_k(D_n)\}_{k \in \mathbb{Z}}$ is called the Bernstein filtration on D_n .

Let M be a finitely generated left D_n -module. A family $\{F_k(M)\}_{k \in \mathbb{Z}}$ of K -subspaces of M is called a Bernstein filtration of M if it satisfies

- (1) $F_k(M) \subset F_{k+1}(M) \quad (\forall k \in \mathbb{Z}), \quad \bigcup_{k \in \mathbb{Z}} F_k(M) = M$
- (2) $F_j(D_n)F_k(M) \subset F_{j+k}(M) \quad (\forall j, k \in \mathbb{Z})$

(3) $F_k(M) = 0$ for $k \ll 0$

Moreover, $\{F_k(M)\}$ is called a good Bernstein filtration if

(4) $F_k(M)$ is finite dimensional over K for any $k \in \mathbb{Z}$.

(5) $F_j(D_n)F_k(M) = F_{j+k}(M)$ ($\forall j \geq 0$) holds for $k \gg 0$.

Let $\{F_k(M)\}$ be a good Bernstein filtration of M . Then there exists a polynomial $h(T) = h_d T^d + h_{d-1} T^{d-1} + \cdots + h_0 \in \mathbb{Q}[T]$ such that

$$\dim_K F_k(M) = h(k) \quad (k \gg 0)$$

and $d!h_d$ is a positive integer. We call $h(T)$ the Hilbert polynomial of M with respect to the filtration $\{F_k(M)\}$. The leading term of $h(T)$ does not depend on the choice of a good Bernstein filtration $\{F_k(M)\}$. The degree d of the Hilbert polynomial $h(T)$ is called the dimension of M and denoted by $\dim M$. The multiplicity of M is defined to be $d!h_d$, which we denote by $\text{mult } M$.

If $M \neq 0$, then the dimension of M is not less than n (Bernstein's inequality). We call M holonomic if $M = 0$ or $\dim M = n$. It is known that $H_I^j(R)$ is holonomic for any ideal I of R and any integer j ([6]).

If M is a holonomic left D_n -module, we have an equality $\text{length } M \leq \text{mult } M$. Moreover, the multiplicity is additive for holonomic left D_n -modules.

Lemma 2.1 *Let $h_0 = h_0(x) \in K[x]$ be a linear polynomial and I be an ideal of $R := K[x]$. Let $R' := R/Rh_0$ be the affine ring associated with the hyperplane $h_0(x) = 0$ and set $I' = (I + Rh_0)/Rh_0$. Then we have*

$$\text{length } H_{I+Rh_0}^i(R) = \text{length } H_{I'}^{i-1}(R'), \quad \text{mult } H_{I+Rh_0}^i(R) = \text{mult } H_{I'}^{i-1}(R')$$

for any integer i .

Proof: Since $H_{Rh_0}^i(R) = 0$ for $i \neq 1$, there is an isomorphism

$$H_{I+Rh_0}^i(R) \cong H_{I+Rh_0}^{i-1}(H_{Rh_0}^1(R)).$$

We may assume by an affine coordinate transformation, which preserves the Bernstein filtration, that $h_0(x) = x_n$. Then we may regard $R' = K[x_1, \dots, x_{n-1}]$ and have an isomorphism

$$H_{Rh_0}^1(R) \cong R' \otimes_K H_{(x_n)}^1(K[x_n]),$$

where the tensor product on the right-hand side is a left module over $D_n = D_{n-1} \otimes_K D_1$ with D_1 being the ring of differential operators in the variable x_n .

Let $\{f_1, \dots, f_r\}$ be a set of generators of I . We may assume that f_1, \dots, f_r belong to R' . Then for $0 \leq i_1 < \dots < i_k \leq r$ with $k \in \mathbb{N}$, the localization by $f_{i_1} \cdots f_{i_k}$ yields

$$(H_{Rh_0}^1(R))_{f_{i_1} \cdots f_{i_k}} = R'_{f_{i_1} \cdots f_{i_k}} \otimes_K H_{(x_n)}^1(K[x_n]).$$

On the other hand, we have

$$(H_{Rh_0}^1(R))_{x_n} = R' \otimes_K (H_{(x_n)}^1(K[x_n]))_{x_n} = 0.$$

Hence $H_{I+Rh_0}^{i-1}(H_{Rh_0}^1(R))$ is the $(i-1)$ -th cohomology group of the Čech complex

$$\begin{aligned} 0 &\longrightarrow R' \otimes_K H_{(x_n)}^1(K[x_n]) \longrightarrow \bigoplus_{1 \leq i \leq r} R'_{f_i} \otimes_K H_{(x_n)}^1(K[x_n]) \\ &\longrightarrow \bigoplus_{1 \leq i_1 < i_2 \leq r} R'_{f_{i_1} f_{i_2}} \otimes_K H_{(x_n)}^1(K[x_n]) \longrightarrow \dots \longrightarrow R'_{f_1 \cdots f_r} \otimes_K H_{(x_n)}^1(K[x_n]) \longrightarrow 0, \end{aligned}$$

which is isomorphic to $H_{I'}^{i-1}(R') \otimes_K H_{(x_n)}^1(K[x_n])$. This implies

$$\begin{aligned} H_{I+Rh_0}^i(R) &\cong H_{I'}^{i-1}(R') \otimes_K H_{(x_n)}^1(K[x_n]) \\ &\cong H_{I'}^{i-1}(R') \otimes_K (D_1/D_1 x_n) \cong (D_n/D_n x_n) \otimes_{D_{n-1}} H_{I'}^{i-1}(R'), \end{aligned}$$

where $D_n/D_n x_n$ is regarded as a (D_n, D_{n-1}) -bimodule. The rightmost term is the D -module theoretic direct image of $H_{I'}^{i-1}(R')$ with respect to the inclusion $H_0 := \{x \in V \mid x_n = 0\} \rightarrow V$. In view of Kashiwara's equivalence in the category of algebraic D -modules (see e.g., Theorem 7.11 of [4] or Theorem 1.6.1 of [5]), there is a one-to-one correspondence between the D_{n-1} -submodules M of $H_{I'}^{i-1}(R')$ and the D_n -submodules $M \otimes_K H_{(x_n)}^1(K[x_n])$ of $H_{I+Rh_0}^i(R)$. This implies

$$\text{length } H_{I+Rh_0}^i(R) = \text{length } H_{I'}^{i-1}(R').$$

Next, let us show

$$\text{mult } H_{I+Rh_0}^i(R) = \text{mult } H_{I'}^{i-1}(R').$$

Let $\{F_k\}$ be a good Bernstein filtration of $H_{I'}^{i-1}(R')$ and set $m = \text{mult } H_{I'}^{i-1}(R')$. We may assume $F_k = 0$ for $k < 0$. Then there exists a polynomial $p(k)$ and $k_1 \in \mathbb{Z}$ such that

$$\dim_K F_k = p(k) = \frac{m}{(n-1)!} k^{n-1} + (\text{terms with degree } < n-1)$$

holds for $k \geq k_1$. Define a filtration $\{G_k\}$ on $H_{I'}^{i-1}(R') \otimes_K H_{(x_n)}^1(K[x_n^{-1}])$ by

$$G_k := \sum_{j=0}^k F_j \otimes_K (K[x_n^{-1}] + \cdots + K[x_n^{-(k-j)-1}]) = \bigoplus_{j=0}^k F_j \otimes_K K[x_n^{-(k-j)-1}].$$

It is easy to see that $\{G_k\}$ is a good Bernstein filtration. Hence we have

$$\dim_K G_k = \sum_{j=0}^k \dim_K F_j = \sum_{j=0}^{k_1-1} \dim_K F_j + \sum_{j=k_1}^k p(j).$$

By the assumption, there exists a polynomial $q(k)$ of degree $\leq n-2$ such that

$$p(j) = \frac{m}{(n-1)!} j(j+1) \cdots (j+n-2) + q(j).$$

Since

$$\sum_{j=k_1}^k j(j+1) \cdots (j+n-2) = \frac{1}{n} \{k(k+1) \cdots (k+n-1) - (k_1-1)k_1 \cdots (k_1+n-2)\},$$

we have

$$\dim_K G_k = \frac{m}{n!} k^n + (\text{terms with degree} < n) \quad (\forall k \geq k_1).$$

Thus we also have $\text{mult } H_{I+Rx_n}^i(R) = m$. This completes the proof. \square

3 Hyperplane arrangements

Let $f \in K[x]$ be a multiple of essentially distinct linear polynomials. Let \mathcal{A} be the hyperplane arrangement in $V := K^n$ defined by f .

Theorem 3.1 *Let H_0 be an element of \mathcal{A} . Set $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and let f' be the product of the defining polynomials of hyperplanes belonging to \mathcal{A}' . Let us regard*

$$\mathcal{A}'' := \{H \cap H_0 \mid H \in \mathcal{A}', H \cap H_0 \neq \emptyset\}$$

as a hyperplane arrangement in the affine space H_0 . Let $R'' = R/Rh_0$ be the affine ring of H_0 , where h_0 is a polynomial of first degree defining H_0 . Let $f'' \in R''$ be the product of the defining polynomials of the elements of \mathcal{A}'' . Then we have

$$\begin{aligned} \text{length } H_{(f)}^1(R) &= \text{length } H_{(f')}^1(R) + \text{length } H_{(f'')}^1(R'') + 1, \\ \text{mult } H_{(f)}^1(R) &= \text{mult } H_{(f')}^1(R) + \text{mult } H_{(f'')}^1(R'') + 1. \end{aligned}$$

Proof: By the Mayer-Vietoris exact sequence, we get an exact sequence

$$0 \longrightarrow H_{(f')}^1(R) \oplus H_{(h_0)}^1(R) \longrightarrow H_{(f)}^1(R) \longrightarrow H_{(f')+(h_0)}^2(R) \longrightarrow 0$$

of holonomic left D_n -modules. Since the length and the multiplicity of $H_{(h_0)}^1(R)$ are both one, it follows that

$$\begin{aligned} \text{length } H_{(f)}^1(R) &= \text{length } H_{(f')}^1(R) + \text{length } H_{(f')+(h_0)}^2(R) + 1, \\ \text{mult } H_{(f)}^1(R) &= \text{mult } H_{(f')}^1(R) + \text{mult } H_{(f')+(h_0)}^2(R) + 1. \end{aligned} \quad (1)$$

Since $(f'') = R''f'' \cong (Rf' + Rh_0)/Rh_0$, Lemma 2.1 implies

$$\begin{aligned} \text{mult } H_{(f')+(h_0)}^2(R) &= \text{mult } H_{(f'')}^1(R''), \\ \text{length } H_{(f')+(h_0)}^2(R) &= \text{length } H_{(f'')}^1(R''). \end{aligned}$$

This completes the proof in view of (1). \square

Corollary 3.1 $\text{length } H_{(f)}^1(R) = \text{mult } H_{(f)}^1(R)$.

Proof: This can be easily proved by induction on $\sharp\mathcal{A}$ by using Theorem 3.1. \square

The intersection poset $L(\mathcal{A})$ is the set of the non-empty intersections of elements of \mathcal{A} . For $X, Y \in L(\mathcal{A})$, define $X \leq Y$ if and only if $X \supset Y$. For $X, Y \in L(\mathcal{A})$, the Möbius function $\mu(X, Y)$ is defined recursively by

$$\mu(X, Y) = \begin{cases} - \sum_{X \leq Z < Y} \mu(X, Z) & \text{if } X < Y \\ 1 & \text{if } X = Y \\ 0 & \text{otherwise} \end{cases}$$

Set $\mu(X) = \mu(V, X)$. Then $(-1)^{\text{codim } X} \mu(X)$ is positive (see e.g. Theorem 2.47 of [7]). The Poincaré polynomial of the arrangement \mathcal{A} is defined by

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) (-t)^{\text{codim } X}.$$

Theorem 3.2

$$\text{length } H_{(f)}^1(R) = \pi(\mathcal{A}, 1) - 1 = \sum_{X \in L(\mathcal{A}) \setminus \{V\}} |\mu(X)|.$$

Proof: Let H_0 be an element of \mathcal{A} defined by a first degree polynomial h_0 . Let us prove the equality by induction on $\#\mathcal{A}$. Since $H_{(h_0)}(R)$ is simple, the equality holds if $\mathcal{A} = \{H_0\}$ with $\pi(\mathcal{A}, 1) = 2$. Let $\mathcal{A}', \mathcal{A}''$ be as in the proof of Theorem 3.1. By the induction hypothesis, we have

$$\text{length } H_{(f')}^1(R) = \pi(\mathcal{A}', 1) - 1, \quad \text{length } H_{(f'')}^1(R'') = \pi(\mathcal{A}'', 1) - 1.$$

Hence by Theorem 3.1 we get

$$\begin{aligned} \text{length } H_{(f)}^1(R) &= \text{length } H_{(f')}^1(R) + \text{length } H_{(f'')}^1(R'') + 1 \\ &= \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1. \end{aligned}$$

On the other hand, $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$ holds (see e.g., Theorem 2.56 of [7]). Thus we get

$$\text{length } H_{(f)}^1(R) = \pi(\mathcal{A}', 1) + \pi(\mathcal{A}'', 1) - 1 = \pi(\mathcal{A}, 1) - 1.$$

This completes the proof. \square

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